Pak. J. Sci. Ind. Res. 2003 ***46*** *(3) 148-150*

PLANETARY ORBITS IN AXISYMMETRIC VACUUM GRAVITATIONAL FIELDS

Muhammad J Iqbal [a]* ***and Jawaid Quamar*** [b]

[a]*Department of Mathematics, University of Karachi, Karachi 75270, Pakistan*

[b]*Institute of Space and Planetary Astrophysics, University of Karachi, Karachi 75270, Pakistan*

(Received 11 November 1999; accepted 14 June 2002)

An exact axisymmetric asymptotically flat field, is determined by Einstein equations, possessing a quadrupole moment due to a static mass, may be treated as a perturbation on Schwarzschild field. Exploiting this, planetary equations under the influence of the mentioned gravitational field has been worked out. The results exhibit features that shed new light on issues in relativistic celestial mechanics and models of planetary motion.

Key words: Planetary orbits, Vacuum, Gravitational field.

Introduction

Celestial mechanics has been the driving force which spurred the great mathematicians to do incredible efforts to find useful methods of analysing planetary motion. The invented elegant tools had astonishing applicability in many diverse fields. Even today there are significant problems asking for their solution and one such problem is presented here. It is an empirical fact that stars, in particular the Sun, are rotating gravitating sources. The gravitational field of such a source may be represented by rotating Kerr space-time or its other useful variants, or space-time such as Tomimatsu-Sato space-time, etc. Metrics of such space-times are axisymmetric. Thus, a planetary theory for such a gravitational field should in principle take into account the possible effect of the rotation of the central mass. To keep mathematics tractable, in the first instance axisymmetric metrics which are static, has been considered here.

The problem of axisymmetic static vacuum field in relativistic gravitodynamics was first formulated by Weyl (1917). As in case of vacuum or electrostatic vacuum, the metric tensor could be diagonalised via an introduction of two (Spatial) functions, λ and μ:

$$ds \otimes ds \equiv e^{2\lambda}\, dt \otimes dt - e^{-(\lambda-\mu)} (dp \otimes dp + dz \otimes dz) + p^2 e^{-2\lambda}\, d\varphi \otimes d\varphi \quad \ldots\ldots (1)$$

Thus the field equations are quite simple for vacuum

$$\lambda_{,11} + \lambda_{,1}/p + \lambda_{,33} = 0, \quad \ldots\ldots (2)$$

$$\mu_{,3} = 2p\lambda_{,1}\lambda_{,3}, \quad \ldots\ldots (3)$$

$$\mu_{,1} = p[(\lambda_{,1})^2 - (\lambda_{,3})^2]. \quad \ldots\ldots (4)$$

*Author for correspondence

Here λ and μ are functions of p and z, these coordinates corresponding to a lorentzian orthonormal cylindrical polar holonomic basis, and commas (as subscripts in the preceding relations) denote as usual partial differentiation with respect to the chosen coordinates. As we see, Eq (2) satisfied by λ is a flat-space laPlace equation, while μ is calculable by quadratures, employing λ. It follows from the fact that (2) is the integrability conditions for the system (3) and (4).

An exact axisymmetric asymptotically flat solution of Einstein equation, possessing a quadrupole moment due to a static mass represents a small deformation of Schawarzschild solution (Hernandez *et al* 1993). It produces a deformation in away that the full spherical symmetry possessed by a gravitating source, like the Sun, decreases. It is not difficult to see that the general asympotically flat solution of Eq (2) is given by:

$$\lambda \equiv \frac{1}{2}\ln\frac{x-1}{x-1} + \frac{5}{8}q(3y^2-1)\left[\left(\frac{3x^2-1}{4} - \frac{1}{3y^2-1}\right)\ln\frac{x-1}{x+1} - \frac{2x}{(x^2-y^2)(3x^2-1)} + \frac{3}{2}x\right], \quad \ldots\ldots (5)$$

where q is a certain parameter. Herein prolate spherical coordinates (x, y) have been introduced:

$$x \equiv \frac{p+r}{2M}, \qquad y \equiv \frac{p-r}{2M} \quad \ldots\ldots (6)$$

with

$$p \equiv [p^2 + (z+M)^2]^{1/2} \quad \ldots\ldots (7)$$

$$r \equiv [p^2 + (z-M)^2]^{1/2} \quad \ldots\ldots (8)$$

The first term in eq. (5) is the Schwarzschild spherically symmetric solution, while other terms describe the deformation of the massive source. An integration of Eqs (3) and (4) in terms of these new coordinates, with λ given by eq. (5), leads to the following value for the function μ:

$$\mu = \frac{1}{2}\left(1 + \frac{1}{6}q^2\right) \mathrm{In}\left(\frac{x^2-1}{x^2-y^2}\right) \frac{1}{4}\left(qx(1-y^2)\right) \quad \text{.........(9)}$$

$$\left[1 - \frac{1}{16} q \left(x^2 + 7y^2 - 9x^2y^2 - \frac{8}{3}\frac{x^2+1}{x^2-y^2}\right)\right]$$

$$+ \frac{1}{256} q^2 (x^2-1)(1-y^2)$$

$$(x^2 + y^2 - 9x^2y^2 - 1)\ \mathrm{In}^2 \frac{x-1}{x+1}$$

$$- \frac{1}{2} q(1-y^2)\left[1 - \frac{1}{32} q\left(x^2 + 4y^2 - 9x^2y^2 + 4\right)\right]$$

$$\frac{1}{12} q^2x^2 \frac{1-y^2}{x^2-y^2} - \frac{1}{6} q(x^2+y^2) \frac{1-y^2}{(x^2-y^2)^2}$$

$$\frac{1}{144} q^2 (2x^6 - x^4 + 3x^4y^2$$

$$- 6x^2y^2 + 4x^2y^4 - y^4 - y^6) \frac{1-y^2}{(x^2-y^2)^4}$$

Relations of eq. (5) and (9) constitute a useful solution of an axisymmetric field in that they can be exploited to look into physical properties associated with issues like multipole moment of the field in question.

Orbits of Test Particles

The lagrangian for our metric becomes

$$L \equiv \frac{1}{2}(e^{2\lambda}\dot{t}^2 - e^{-(\lambda-\mu)}(\dot{p}^2 + \dot{z}^2) + p^2 e^{-2\lambda}\dot{\varphi}^2] \quad \text{..............(10)}$$

So the Euler-Lagrange equations are

$$\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \left(\frac{\partial L}{\partial x^i}\right) = O, \qquad (i \in \{0, 1, 2, 3\}) \quad \text{..........(11)}$$

Here a dot denotes a differentiation with respect to the affine parameter u. By eq.(10) in (11), we obtain the following equations of motion of the test particles moving in an axisymmetric static vacuum gravitational field:

$$\ddot{t} + 2\lambda_{,1}\dot{p}\dot{t} + 2\lambda_{,3}\dot{z}\dot{t} = 0, \quad \text{....................................(12)}$$

$$\ddot{p} - (\lambda_{,3} - \mu_{,1})\dot{p}^2 - 2(\lambda_{,3} - \mu_{,3})\dot{p}\dot{z} \quad \text{........................(13)}$$

$$+ (\lambda_{,1} - \mu_{,1})\dot{z}^2 e^{2(2\lambda-\mu)}\lambda_{,1} t^2$$

$$+ pe^{-2\mu}(p\lambda_{,1} - 1)\dot{\varphi}^2 = 0,$$

$$\ddot{z} - (\lambda_{,3} - \mu_{,3})\dot{z}^2 - 2(\lambda_{,1} - \mu_{,1})\dot{p}\dot{z} \quad \text{.......................(14)}$$

$$- (\lambda_{,3} - \mu_{,3})\dot{p}^2 + e^{2(2\lambda-\mu)}\lambda_{,3}\dot{t}^2$$

$$+ p^2 e^{-2\mu}\lambda_{,3}\dot{\varphi}^2 = 0,$$

$$\ddot{\varphi} + 2\left(\frac{1}{p} - \lambda_{,1}\right)\dot{p}\dot{\varphi} - \lambda_{,3}\dot{z}\dot{\varphi} = 0, \quad \text{..................(15)}$$

One can easily deduce the following equation from (14):

$$p^2 \frac{d\varphi}{ds} = e^{2\lambda} h, \qquad u = s, \quad \text{..(16)}$$

Where h is a constant of integration. Substituting the value of λ from (5) in eq. (16), we get

$$p^2 \frac{d\varphi}{ds} = h.\sqrt{1 - \frac{2}{x+1}}$$

$$+ h\exp\left\{\frac{5}{8} q(3y^2 - 1)\left[\left(\frac{3x^2-1}{4} - \frac{1}{3y^2-1}\right)\right.\right.$$

$$\left.\left.\mathrm{In}\frac{x-1}{x+1} - \frac{2x}{(x^2-y^2)(3x^2-1)} + \frac{3}{2}x\right]\right\}$$

$$= h\left[\left\{-\frac{1}{2} + 4a + 0.86\exp(0.76q)\right\}\right.$$

$$\left. + \{2 - 2a - 0.43\exp(0.76q)\}x - (4+a)x^2 + by^2 + \cdots\right], \text{..(17)}$$

Where a and b are certain constants which can be evaluated. The above equation shows that areal velocity is not, in general, constant in an axisymmetric static vacuum field – in contrast to the case of Schwarzschild space-time. Empirically, however, to a high degree of accuracy the areal velocity is constant for our solar system planets. In our case, areal velocity becomes constant if we set $\lambda = 0$. But then metric (1) would reduce to a flat (minkowskian) space time. It follows that, of necessity, $\lambda \neq 0$. This suggests if we incorporate the contribution of rotation of the central gravitating body in a planetary theory, a residual slight perturbation on the standard constant areal velocity should exist. At the moment, we are looking into this possibility. Clearly, this finding may ultimately shed new light on centuries-old celestial mechanics based on Keplerian laws.

It is known, with the exception of the two-body motion, that the problems of celestial mechanics are generally incapabable of exact mathematical solution. Due to this difficulty (of absence of an exact solution to three-body and, generally, n-body problem), one often resorts to or tries to exploit the method of two-body problem . This is particularly true of standard Galilie-Newtonian theory itself. This same difficulty is perhaps also responsible for the popular misconception that planets of our solar system have constant areal velocity. In fact, however, staying right in Galilie-Newtonioan theory, if one switches from the two-body problem to even the restricted three-body problem, areal velocity turns out to be nonconstant in general (Lagrange 1772; Jacobi 1966; Bartin 1987). This situation nicely compares with our finding of the general nonconstancy of the areal velocity in relativistic celestial mechanics.

Also, a straightforward manipulation of eq. (14) shows that the orbit equation in the polar plane in case of an axisymmetric static vacuum field is:

$$\frac{d^2 u}{d\varphi^2} + e^{-2\mu} u = 3e^{2(\lambda-\mu)} \frac{d\lambda}{du} + 2h^2 e^{-2\mu} \frac{d\lambda}{du} u^2 \text{.........(18)}$$
$$+ \left(\frac{d\mu^2}{du} - 2\frac{d\lambda}{du} \right) \left(\frac{du}{d\varphi} \right).$$

With $u \equiv 1/p$. We are looking into further implications of the formalism given above for which study is under way.

Conclusion

As we have seen, areal velocity is, in general, not constant for space-times with an azimuthal Killing 1-vector field. However, if we calculate the areal velocity at the origin of prolate coordinates that we have introduced, we are left with only the first term in the expansion (eq.18). Thus, the following classical Keplerian result, in Galilei-Newtonian physics, of the constancy of areal velocity is recovered. This particular case of our general result is also corroborated empirically as we know that to a high degree of accuracy the areal velocity is constant for our solar system planets.

It seems, nevertheless, that azimuthal symmetry of space time structure plays further and as yet generally unknown role in the theory of relativistic celestial mechanics. The presence of extra terms in eq. (17) may be interpreted as a physical explanation of the observed smallness of the rotation of stars, in particular of the Sun. Accordingly, any possible oblateness arising from a rotation of the sun should be small. This fact is particularly significant for gravity theories should in principle lead back to Einstein's original theory.

Another aspect of our result follows with the earlier work which argued that due to the solar rotation the perihelion (Quamar 1986), advances by an amount of $3m^2 \omega^2/h^4$ during each revolution of the test body. However, this solar rotation effect is small enough to be observed and detected within the framework of the technology-2000 A.D. Finding of this paper, thus, also confirms the result obtained earlier (Quamar 1986).

References

Bartin R H 1987 *An introduction to the mathematics and method of astrodynamics,* AIAA Education Series.

Hernandez-Pastora J L, Martin J 1993 *Classical and Quantum Gravity,* 2581-2585.

Jacobi C G J 1866 Vorlesungen ü ber dynamik, (posthumous).

Lagrange 1772 Essai sur le probleme des trois, Paris Academy.

Quamar J 1986 *Riazi,* **8** 27-31.