Short Communication

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On Generalization of Banach's Fixed Point Theorem

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In this paper we have shown that if T be a continuous self-mapping of a metric space (x, d) such that (i) $d(Tx, Ty) < K_1d(x, Tx) + K_2d(y, Ty) + k_3d(x, y)$ for each $x, y \in x$ $k_1, k_2, k_3 > 0$, $k_1+k_2+k_3 < 1$, (ii) there exist a subset $M \subset x$ and a point $x_0 \in M$ such that $d(x, x_0) - d(x_0, Tx) \geq kd(x_0, Tx_0)$ for every $x \in x\text{-}M$, $k = k_1/1\text{-}k_2\text{-}k_3$ and (iii) T maps M into a compact subset of x. Then T has a unique fixed point in M.

A mapping T of a metric space (x, d) into itself is called a contraction mapping if the condition $d(T(a), T(b)) \leq \lambda d(a, b)$ with the constant λ, $0 \leq \lambda < 1$, holds for every $a, b \in x$.

The most famous Banach's contraction principle (Banach 1992) states that a contraction mapping on a complete metric space (x, d) has a unique fixed point.

A mapping T of a metric space (x, d) into itself is called globally contractive if the condition $d(T(x), T(y)) < \lambda d(x, y)$ with constant λ, $0 \leq \lambda < 1$, holds for every $x, y \in x$, $x \neq y$.

Rakotch (Rakotch 1962) has generalized the Banach's contraction principle by replacing λ with a function $\lambda(x, y)$ by suitably defining the family $\{\lambda(x, y)\}$ of functions.

The paper contains the idea of the generalized contraction mapping as introduced by Rakotch (1962) and a theorem on sequence of mappings, and common fixed point theorems have been proved. Throughout this paper, (x, d) will denote a metric space (unless otherwise stated).

Definition 1.1. Let F denote the family of functions $\lambda(x, y)$ satisfying the following conditions: (i) $\lambda(x, y) = \lambda(d(x, y)$, i.e., λ depends on the distance between x and y only, (ii) $0 \leq \lambda(d(x, y) < 1$ for every $d(x, y) > 0$, (iii) $\lambda(d(x, y)$ is a monotonically decreasing function of d(x, y).

Theorem 2.1. Let T be a continuous mapping of x into itself such that

$$d(Tx, Ty) < \alpha d(x, Tx) + \beta d(y, Ty) + \gamma d(x, y) \text{ for } x, y \in x, x \neq y, \alpha, \beta, \gamma > 0, \alpha+\beta+\gamma < 1.$$

If for some $x_0 \in x$ the sequence $\{T^n(x_0)\}$ has a subsequence $\{T^{n_k}(x_0)\}$ converging to a point $u \in x$, then u is a unique fixed point of T.

Proof. Let $x_0 \in x$ be arbitrary and let us define the sequence $\{x_n\}$ of elements as $x_n = T^n(x_0)$, $x_{n+1} = T(x_n)$, n = 0, 1, 2...... It can be easily seen that the sequence $\{x_n\}$ also converges to a point u in x, i.e. $\lim_{n\to\infty} x_n = u$.

Since T is continuous, $T(u) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = u$.
If possible, let u and v be two fixed points in x such that $u \neq v$. So, $d(u,v) > 0$.

Now we have

$$d(u, v) = d(T(u), T(v) < \alpha d(u, T(u) + \beta d(v, T(v) + \gamma d(u, v)$$
$$= \gamma d(u, v)$$

which is impossible. Hence the fixed point u is unique.

Theorem 2.2. Let T be a continuous mapping of x into itself such that

$d(T(x), T(y) < \alpha d(x, T(x) + \beta d(y, T(y) + \gamma d(x, y)$ for $x, y \in x$, $x \neq y$, $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma < 1$. Further suppose that there exist a subset $M \subset x$ and a point $x_0 \in M$ such that

$$d(x, x_0) - d(x_0, Tx) \geq kd(x_0, Tx_0)$$

for every $x \in x\text{-}M$, where $k = \alpha/1\text{-}\beta\text{-}\gamma < 1$ and that T maps M into a compact subset of x.

Then there exists a unique fixed point of T.

Proof. Let $x_0 \neq T(x_0)$ and define $x_n = T^n(x_0)$, $x_{n+1} = T(x_n)$, n=0, 1, 2......... Since T maps M into a compact set, we shall show that $x_n \in M$ for every positive integer n and the rest of the proof will follow as a direct consequence of Theorem 2.1.

It has been noted in Theorem 2.1 that $\{x_n\}$ is a Cauchy sequence in M and it is easy to see that

$$d(x_n, x_{n+1}) < (\alpha/1\text{-}\beta\text{-}\gamma)^n d(x_0, T(x_0)$$
$$< (\alpha/1\text{-}\beta\text{-}\gamma) d(x_0, T(x_0)$$

So we have

$$d(x_n, x_0) \leq d(x_0, x_1) + d(x_1, x_{n+1}) + d(x_n, x_{n+1})$$
$$< d(x_0, T(x_0) + d(T(x_0), T(x_n) + (\alpha/1\text{-}\beta\text{-}\gamma) d(x_0, T(x_0)$$

Therefore

$$d(x_n, x_0) \leq \{d(x_0, T(x_0) + d(T(x_0), T(x_n)\} < kd(x_0, T(x_0)$$

i.e. $d(x_n, x_0) - d(x_0, T(x_n) < kd(x_0, T(x_0)$, where $k= (\alpha/1- \beta - \gamma)$

and so from (*) it follows that $x_n \in M$ for every n.

As a direct consequence of Theorem 2.2, we have the following:

Corollary. Let T be a continuous mapping of x into itself satisfying the following condition:

$$d(T(x), T(y) < k_1 d(x, T(x) + k_2 d(y, T(y) + k_3 d(x, y)$$

$k_1, k_2, k_3 > 0$ and $k_1 + k_2 + k_3 < 1$, for every $x, y \in x, x \neq y$. Further suppose that there exists a point $x_0 \in x$ such that

$D(x_0, T(x) \leq \alpha(x, x_0)\, d(x, x_0)$ for every $x \in x$, where $\alpha(x, x_0) \in F$ and that T maps $S(x_0, r) = \{x\ d(x, x_0) < r\}$ with

$$r = \frac{k\, d(x_0, T(x_0)}{1 - (\alpha+\beta)\, [kd(x_0, T(x_0)]}, \qquad k=k_1/1-k_2-k_3$$

into compact subset of x.

Then there exists a unique fixed point of T.

Proof. In Theorem 2.2, let us take $M=S(x_0, r)$. Then since $\alpha(x, x_0) = \alpha(d(x, x_0) \in F$ and $\beta(x, x_0) = \beta(d(x, x_0) \in F$, $\alpha(d)$, $\beta(d)$ are monotone decreasing and, since $r \geq kd(x_0, T(x_0)$, we have [from the given condition]

$$d(x, x_0) - d(x_0, T(x) \geq d(x, x_0) - \alpha(x, x_0)\, d(x, x_0) - \beta(x, x_0)\, d(x, x_0)$$
$$= [1 - (\alpha(x, x_0) + \beta(x, x_0)]\ d(x, x_0)$$

If $d(x, x_0) \geq r$, i.e., $x \notin M$, then we have

$$d(x, x_0) - d(x_0, T(x) \geq [1-(\alpha+\beta)\ \{kd(x_0, T(x_0)\}]\ r$$
$$= k\, d(x_0, T(x_0)$$

So the condition (*) of Theorem 2.2 holds. Hence this corollary follows.

Key words: Banach, Fixed point, Self mapping.

References

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